A Generalized Velocity Distribution for Non-Newtonian Fluids

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The universal velocity profile equation for turbulent flow is obtained for the power law approximation equation. The profile equation should be valid from wall to the center line, from laminar to turbulent flow, and for Newtonian and non-Newtonian power law materials. The equation is compared with other theoretical and experimental results. In the center line region, where the other theoretical development is valid, both equations give essentially the same results. The experimental confirmation with literature data is good over the entire flow regime.

The desirability of a single equation for the description of the turbulent flow velocity profile has long been recognized. The closest approach to this goal has been the semiempirical method of Pai (*7, 8, 9*). The work of this paper is to extend this method to non-Newtonian fluids. A universal velocity profile equation for turbulent flow, valid from wall to center line, is obtained for the power law approximation equation. This equation is compared with other theoretical and experimental developments.

Little need be said about the velocity distribution in laminar flow. For the ideal plastic the distributions are given in references *1* and *3*. For the power law fluid

$$v/v_{avg} = [(1+3n)/(1+n)] [1-(r/r_0)^{(n+1)/n}] \quad (1)$$

or

$$v/v_{max} = 1-(r/r_0)^{(n+1)/n} \quad (2)$$

These equations have had excellent experimental confirmation by Richardson, McGinnis, and Beatty (*4*).

The analysis of turbulent velocity distribution has been considered only recently (*2, 5, 6*). In the present work the Newtonian approach of Pai (*1, 7, 8, 9*) will be extended to non-Newtonian power law fluids. This method gives one equation which is valid from wall to center line. The resulting equation can be compared with experimental data (*5, 6*) and with a theoretical derivation based on the log distribution (*2*).

The desired equation can be obtained from the Reynolds equations for turbulent flow (*1, 7, 8*); however the following method gives the same result more directly. From an integration of the Reynolds equations it can be shown that the total shear is composed of a laminar and a turbulent component:

$$\tau = \tau_L + \tau_T \quad (3)$$

For the power-law fluid

$$\tau_L = K(-dv/dr)^n \quad (4)$$

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and in general

$$\tau_T = \rho\, \overline{v'v'_r} \quad (5)$$

Combining these two equations with Equation (3), and using the linear variation of τ across a pipe one gets

$$\tau/\rho = (K/\rho)(-dv/dr)^n + \overline{v'v'_r} = (r/r_0)(\tau_w/\rho) = (r/r_0)U^{*2} \quad (6)$$

If for Equation (4) one uses Newtons' law ($n = 1, K = \mu$), the resulting equation [analogous to (6)] is identical to the equation obtained by an integration of the Reynolds equation for pipe or channel flow (*1, 7, 8*). For nonpower law materials (or for a more rational approach) other alternate equations could be used to replace Equation (4).

The velocity across the pipe is assumed to be represented by a three term power series of the form

$$v/v_{max} = a_1(r/r_0)^{(n+1)/n} + a_2(r/r_0)^{2m} \quad (7)$$

Only three terms are used because the number of known boundary conditions will allow only the coefficients of three terms to be completely determined.

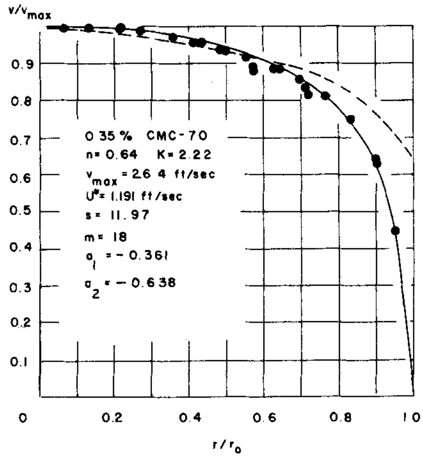

Fig. 1. Distribution for case where integrated average checked both the over-all and theoretical averages. Solid line is Equation (11); dashed line is Equation (15) and data from Shaver (6).

The power of the first term is set because of the known limiting laminar case [Equation (2)]. The constant m is an integer greater than 2 and is set by the experimental data. These restrictions are analogous to those used by Pai. The major requirement is symmetry of the velocity profile in cartesian coordinates (channel flow with $y = 0$ at the center line). For Newtonian materials $(n+1)/n = 2$ and the symmetry of the first term is assured as (y/y_0) changes sign; however for non-Newtonians the symmetry must be assumed. That is $(y/y_0)^{(n+1)/n}$ is assumed always positive. In cylindrical coordinates the problem is not as apparent as (r/r_0) is always positive. m is taken as an integer greater than 2 in order to avoid the same nonsymmetry problem in cartesian coordinates. The symmetry problem associated with the use of the nonrational power law could be avoided by the use of a more rational representation of the shear stress-shear rate equation (Equation (4)].

The coefficients a_1 and a_2 can be obtained as follows.

Since $v = 0$ at $r = r_0$, Equation (7) gives

$$1 + a_1 + a_2 = 0 \quad (8)$$

From Equation (6), when $r = r_0$, $\overline{v'v'_r} = 0$, and

$$(dv/dr)_w = -(\rho U^{*2}/K)^{1/n} \quad (9)$$

The derivative of Equation (7), written at the wall, can be combined with Equation (9) to give

$$-(\rho U^{*2}/K)^{1/n}(r_0/v_{max}) = [(n+1)/n]a_1 + 2m\, a_2 \quad (10)$$

Equations (8) and (10) can be solved for a_1 and a_2 and then combined with Equation (7) to give

$$v/v_{max} = 1 + \left[\frac{s-m}{m-(n+1)/2n}\right](r/r_0)^{(n+1)/n} + \left[\frac{(n+1)/2n - s}{m-(n+1)/2n}\right](r/r_0)^{2m} \quad (11)$$

where

$$s = (\rho U^{*2}/K)^{1/n}(r_0/2v_{max}) = (y_0^+)^{1/n}/2u_0^+ \quad (12)$$

Table 1

Material	Figure	U^* Shaver	U^* Dodge and Metzner	v_{avg} Experimental Over-all	v_{avg} Experimental Integration of distribution	v_{avg} Equation (*34*)
0.35% CMC	1	0.933	1.191	21.3	21.3	21.3
0.41% CMC	—	1.115	1.587	31.1	30.2	31.1
0.62% alginate	2	1.575	1.677	29.7	30.6	29.4

in which

$$y_0^+ = r_0^n U^{*2-n} \rho / K \text{ and } u_0^+ = v_{max}/U^* \quad (13)$$

Equation (11) is valid for power law fluids and gives the velocity distribution over the entire flow area.

Equation (11) can be integrated to give the average velocity across the pipe. The volumetric flow rate is

$$Q = v_{avg}\, \pi\, r_0^2 = 2 \int_0^{r_0} \pi\, v\, r\, dr$$

or

$$v_{avg} = (2/r_0^2) \int_0^{r_0} v\, r\, dr$$

Combining this with Equation (11) and integrating one gets

$$v_{avg}/v_{max} = 1 + 2a_1 n/(3n + 1) + a_2/(m + 1) \quad (14)$$

For the Newtonian laminar flow limit $s = 1$ and $n = 1$, which gives $a_1 = -1$ and $a_2 = 0$. Equation (14) reduces to $v_{avg}/v_{max} = 1/2$.

Unfortunately little non-Newtonian turbulent distribution data is available, and none has been taken very close to the wall. However comparison can be made with the data of Shaver and Merrill (*5, 6*), and an equation of the log distribution type (valid only away from the wall) presented by Dodge and Metzner (*2*):

$$(v_{max} - v)/U^* = 5.66(n)^{0.25} \log [r_0/(r_0 - r)] \quad (15)$$

Two comparisons are given in Figures 1 and 2. The value for m was taken as the integer closest to 1.5 [suggested by Pai (*9*) and based upon Laufer's data, which was taken on air]. (This data extended to the wall and provided information for a critical evaluation of m.) It was found that for the determination of s from Equation (12) the values of U^* reported by Shaver gave much poorer results than those estimated from his values of $N_{Re}(\mu ap)$ and the Dodge and Metzner (*2*) correlation for friction factor. These values of U^* are given on the figures and are compared with Shaver's in Table 1. The average flow rate was measured independently and thus could be used as an independent check on the velocity distribution. The values of the flow rate were obtained by a back calculation from Shaver's values of $N_{Re}(\mu ap)$. The average was also obtained by integration of the velocity distribution curve. These two averages are compared (Table 1) with the calculated average from Equation (14).

The comparison shows that the average velocity from the over-all flow checks better with Equation 14 than with the average from the experimental distribution. In the 0.35% carboxy methyl cellulose run the averages are the same, and Figure 1 shows an almost exact check of Equation (11) with the experimental distribution. Since the theoretical average checks the over-all experimental average on the other two runs, it appears that the experimental distribution data for these two runs may be in error and should be closer to the theoretical line (for example see Figure 2). The deviation is mainly in the wall area.

The agreement between experiment and prediction for the non-Newtonian turbulent velocity profile and the average velocity is satisfactory and strongly supports the adequacy of the assumed profile of Equation (7). In the limit of laminar flow the turbulent velocity profile Equation (11, 12, 13) reduces to the laminar profile Equation (2). Further, in the limit of a Newtonian fluid the equation reduces to the corresponding equation presented by Pai.

Equation (11), based on a series form for the velocity profile, is valid from the wall to the center line, from laminar to turbulent flow, and for Newtonian and non-Newtonian power law materials. The experimental confirmation is good.

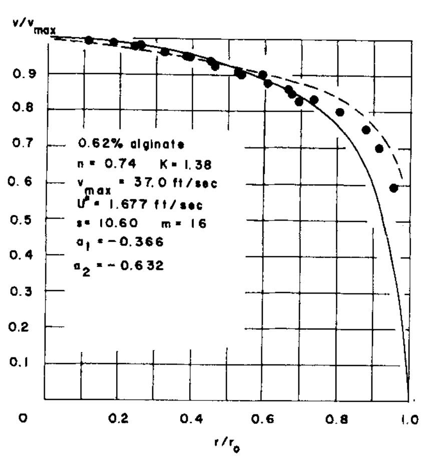

Fig. 2. Distribution for case where the integrated average did not check the over-all or theoretical averages. Solid line is Equation (11); dashed line is Equation (15) and data from Shaver (6).

NOTATION

a_1, a_2 = constants
K = defined by Equation (4)
m = empirical constant
n = defined by Equation (4)
$N_{Re}(\mu ap)$ = $Dv_{avg}\rho/\mu_a p$ based on the apparent viscosity obtained from capillary data, μap
Q = volumetric flow rate
r = radius
r_0 = pipe radius
s = defined by Equation (12)
U^* = $U^* = \sqrt{\tau_w/\rho}$, friction velocity
v = velocity, average at a point
v' = velocity fluctuating component
v_{avg} = velocity, average over pipe
v_{max} = velocity, maximum
v'_r = velocity, radial fluctuating component
$\overline{v'v'_r}$ = turbulent velocity correlation
y_0^+ = defined by Equation (13)
u_0^+ = defined by Equation (13)

Greek Letters

τ = shear stress
τ_w = wall shear stress
τ_L = laminar shear stress
τ_T = turbulent shear stress
π = 3.1416
μap = apparent viscosity as measured on a capillary shear diagram ($4Q/\pi r_0^3$ vs. $r_0\, \Delta p/2L$)
ρ = density

Subscripts

w = wall

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